

**Read these instructions:**

- Leaving the testing room results in a new exam given for the unfinished problems.
- Two detached sheets of notes allowed.
- No electronics.
- Raise your hand for questions or more paper.
- You may leave answers in terms of combinations, permutations, factorials, exponentiation,  $\times$ ,  $\div$ ,  $+$ , and  $-$  of numbers. For instance,  $C(12, 7)/5! - P(9, 2) \times 5! + 7^3$  is an acceptable answer.
- Box your final answer to each problem.

**Curve.**

If  $x$  is your raw grade and  $y$  is the curved grade, then

$$y = \begin{cases} (4/7)x + 60 & \text{if } x \geq 70 \\ (6/7)x + 40 & \text{if } 35 < x < 70 \\ 2x & \text{if } x \leq 35 \end{cases}$$

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**Problem 1.** How many ways can one gold medal, one silver medal, and one bronze medal be awarded to 21 runners in a race? (Assume that there are no ties.)

**Answer:**

$P(21, 3) = 21 \times 20 \times 19$  or 7980 if you choose to simplify.

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**Problem 2A.** How many ways can you rearrange the letters in the word "RECURRENCE"?

**Answer:**

This word has 10 letters, 3 R's, 3 E's, 2 C's so the number of rearrangements is  $\frac{10!}{3!3!2!}$  or 50400 if you choose to simplify.

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**Problem 2B.** How many rearrangements of the letters in the word "RECURRENCE" start with an R **and** end with an R? Examples: "RCEURENCER", "RCREEENUCR", "RCENUREECCR".

**Answer:**

Since we have reserved two of the R's to be at the front and at the end, we are left with "ECURENCE" to permute. This word has 8 letters, 1 R's, 3 E's, 2 C's so the number of rearrangements is  $\frac{8!}{1!3!2!}$  or 3360 if you choose to simplify.

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**Problem 2C.** How many rearrangements of the letters in the word "RECURRENCE" start with a C **and** end with a C? Examples: "CREURRENEC", "CRRREEENUC", "CENUREREC".

**Answer:**

Since we have reserved two of the C's to be at the front and at the end, we are left with "REURRENE" to permute. This word has 8 letters, 3 R's, 3 E's, 0 C's so the number of rearrangements is  $\frac{8!}{3!3!}$  or 1120 if you choose to simplify.

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**Problem 2D.** How many rearrangements of the letters in the word "RECURRENCE" start with a C **or** end with a C (or possibly both)? Examples: "CREURRENCE", "CRRREEENUC", "RCENUREREC".

**Answer:**

- If it starts with a C:
- If it ends with a C: the count is the same as starting with a C.
- If it starts and ends with a C: this was your answer from **Problem 2C**.
- So the number of permutations of "RECURRENCE" that start with a C **or** end with a C (or possibly both) is given by inclusion-exclusion:

$$\frac{9!}{3!3!1!} + \frac{9!}{3!3!1!} - \frac{8!}{3!3!}$$


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**Problem 3.** A UI designer must pick 3 different colors and 2 different fonts from 10 brand-approved colors and 6 brand-approved fonts to use. How many different color and font combinations are possible overall?

**Answer:**

From 10 colors we choose 3, and from 6 fonts we choose 2, so the number of possible choices is:

$$C(10, 3) \times C(6, 2)$$


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**Problem 4A.** How many 3-element subsets of the set  $U = \{a, b, c, d, e, f, g, h, i, j\}$  are there?

**Answer:**

From 10 elements we must choose 3 elements, so there are  $C(10, 3)$  possibilities.

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**Problem 4B.** How many subsets of the set  $U = \{a, b, c, d, e, f, g, h, i, j\}$  have at least two elements?

**Answer 1:**

It is faster to count the complement, or the number of subsets that have at most 1 elements. The answer is:

$$\begin{aligned} & \# \text{ of subsets of } U - \# \text{ of 0-element subsets of } U \\ & \quad - \# \text{ of 1-element subsets of } U \\ & = 2^{10} - 1 - 10 \end{aligned}$$

since there is exactly one 0-element subset (the empty set), and there are 10 elements in  $U$  to choose from to make our 1-element subset.

**Answer 2:**

We can count how many subsets have exactly 2 elements, plus how many subsets have exactly 3 elements, ...

how many subsets have exactly 10 elements, so the total is:

$$C(10, 2) + C(10, 3) + C(10, 4) + C(10, 5) + C(10, 6) + C(10, 7) + C(10, 8) + C(10, 9) + C(10, 10)$$

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**Problem 5A.** A 4-digit PIN (personal identification number) uses digits 0–9. How many 4-digit PINs have no even digits? Examples: "1135", "1357", "5595".

**Answer:**

This is the same as saying the PIN only uses the odd digits of 1, 3, 5, 7, 9 only.

- For the first digit, we have 5 such choices.
- For the 2nd digit, we have 5 such choices.
- For the 3rd digit, we have 5 such choices.
- For the 4th digit, we have 5 such choices.

Overall there are  $5 \times 5 \times 5 \times 5 = 5^4$  choices.

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**Problem 5B.** A 4-digit PIN uses digits 0–9. How many 4-digit PINs have repeated digits? Examples: "1123", "5555", "5253".

**Answer:**

It is faster to count the complement:

$$\begin{aligned} & \# \text{ of 4-digit PINs} - \# \text{ of 4-digit PINs with no repeated digits} \\ & = 10^4 - P(10, 4) \end{aligned}$$

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**Problem 6.** You have 5 different math books and 3 different art books. In how many ways can you arrange these books along a shelf if all the math books must stay together?

**Answer:**

Since the math books must stay together, we can think of the math books as being in a boxed set, so we first rearrange 3 different art books together with the boxed set of math books for a total of  $4!$  permutations of the word  $A_1A_2A_3B$  where  $A_1$  is the first art book,  $A_2$  is the 2nd art book,  $A_3$  is the 3rd art book, and  $B$  is the box that contains the math books.

Next there are  $5!$  ways to rearrange the 5 different math books within that boxed set.

So overall there are  $4! \times 5!$  ways to rearrange these books along a shelf, given that all the math books must stay together.

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**Problem 7A.** How many 5-card hands drawn from a standard deck of 52 playing cards have exactly 2 red cards?

**Answer:**

To make such a 5-card hand, we must draw from the 26 red cards and choose 2 red cards, and from the remaining 26 black cards and choose 3 black cards. This gives a total of

$$C(26, 2) \times C(26, 3)$$

possibilities.

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**Problem 7B.** Draw 5 cards from a standard 52-card deck. How many 5-card hands have at least one Ace?

**Answer:**

It is easier to count the complement:

- Overall, there are  $C(52, 5)$  number of 5-card hands possible.
  - Of these, the number of 5-card hands that have no aces means that we must draw from the 48 non-aces and take 5 cards, or  $C(48, 5)$ .
  - So the number of 5-card hands with at least one Ace is given by  $C(52, 5) - C(48, 5)$ .
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**Problem 8A.** The front row at a concert has 5 seats. How many different seating charts are there that assign 5 different people into these 5 seats? (The people's names are placed on the seats to reserve their seats.)

**Answer:**

- The 1st person has 5 options for seats.
  - The 2nd person has 4 remaining options for seats.
  - The 3rd person has 3 remaining options for seats.
  - The 4th person has 2 remaining options for seats.
  - The 5th person has 1 remaining options for seats.
  - So the answer is  $P(5, 5) = 5!$  possibilities.
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**Problem 8B.** The front row at a concert has 10 seats. How many different seating charts are there that assign 5 different people into these 10 seats? (The people's names are placed on 5 of the 10 seats to reserve their seats.)

**Answer:**

- The 1st person has 10 options for seats.
  - The 2nd person has 9 remaining options for seats.
  - The 3rd person has 8 remaining options for seats.
  - The 4th person has 7 remaining options for seats.
  - The 5th person has 6 remaining options for seats.
  - So the answer is  $P(10, 5) = 10 \times 9 \times 8 \times 7 \times 6$  possibilities.
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**Problem 9.**  $A$  and  $B$  are sets within the universe  $U$  with  $|A| = 10$  and  $|A \cap B| = 6$ ,  $|U| = 20$ , and  $|A \cup B| = 15$ .

**Part 9A.** Find  $|B|$ .

**Part 9B.** Find  $|\sim B|$ .

**Part 9C.** Find  $|\sim A|$ .

**Part 9D.** Find  $|A - B|$ .

**Part 9E.** Find  $|B - A|$ .

**Answer:**

For **Part 9A.**, we apply the inclusion-exclusion formula  $|A \cup B| = |A| + |B| - |A \cap B|$  to get  $15 = 10 + |B| - 6$  so  $|B| = 11$ .

For **Part 9B.**,  $|\sim B| = |U| - |B| = 20 - 11 = 9$ .

For **Part 9C.**,  $|\sim A| = |U| - |A| = 20 - 10 = 10$ .

For **Part 9D.**,  $|A - B| = |A| - |A \cap B| = 10 - 6 = 4$ .

For **Part 9E.**,  $|B - A| = |B| - |A \cap B| = 11 - 6 = 5$ .

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**Problem 10.** A marketing script scraped **500** posts from a tech conference.

- **200** posts use the hashtag **#AI**.
- **250** posts use the hashtag **#Robotics**.
- **100** posts use **neither**.

**Part 10A.** How many scraped posts mention both of these topics?

**Part 10B.** How many scraped posts mention **#AI** but not **#Robotics**?

**Part 10C.** How many scraped posts mention **#Robotics** but not **#AI**?

**Part 10D.** How many scraped posts mention exactly one of these topics?

**Answer:**

Let  $U$  be the universe of the 500 scraped posts. Let  $A$  be the set of posts tagged as AI, and  $R$  the posts tagged with Robotics.

Since  $100 = |\sim(A \cup R)|$ , we know that  $|A \cup R| = 500 - 100 = 400$ .

From inclusion-exclusion, we know  $|A \cup R| = |A| + |R| - |A \cap R|$  or  $400 = 200 + 250 - |A \cap R|$  so  $|A \cap R| = 50$ , which answers **Part 10A**.

Therefore there are  $|A| - |A \cap R| = 200 - 50 = 150$  posts mentioning AI but not robotics, which answers **Part 10B**.

Likewise, there are  $|R| - |A \cap R| = 250 - 50 = 200$  posts mentioning robotics but not AI, which answers **Part 10C**.

Add **Part 10B**. and **Part 10C**. to get the number of posts which mention exactly one of the topics AI or robotics, but not both: **Part 10D**.'s answer is  $150 + 200 = 350$ .

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**Problem 11.** In a league with  $n$  teams, every team plays every other team exactly once. If exactly 55 games are played, what is  $n$ ?

**Answer:**

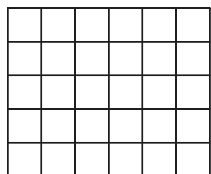
The number of games is the number of pairs of teams chosen from  $n$  teams, or  $C(n, 2) = \frac{n(n-1)}{2}$ . We are given that this number is equal to 55, so we set  $\frac{n(n-1)}{2} = 55$  or  $n(n-1) = 110$ . A few guesses for  $n$  give  $n = 11$ .

(To find  $n$  without guessing, solve the quadratic equation.)

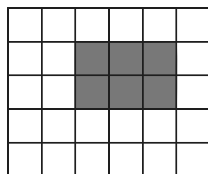
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**Problem 12.** Determine the number of rectangles whose edges lie completely on the grid lines in the following grid (left). Two examples of such rectangles are shown (middle, right).

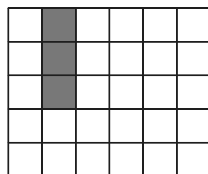
The Grid



Example 1



Example 2

**Answer:**

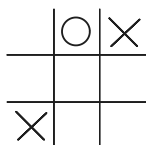
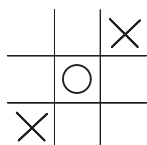
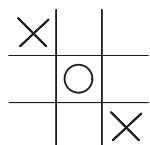
A rectangle is determined by a choice of two  $x$ -coordinates to determine its width and  $x$ -positions, and a choice of two  $y$ -coordinates to determine its height and  $y$ -positions. There are 7 lines to choose two  $x$ 's and 6 lines to choose 2  $y$ 's, for a total of

$$C(7, 2) \times C(6, 2)$$

possibilities.

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**Problem 13.** Tic-tac-toe is a game where two players each alternately place X and O markers onto un-utilized squares of a  $3 \times 3$  grid until certain winning configurations are reached. How many different configurations of the tic-tac-toe board exist where two X's and one O have been placed? Below shows three different valid configurations.

**Answer 1:**

Initially there are 9 empty squares to choose from to place 2 X's, which gives  $C(9, 2)$  options. Next, there are 7 remaining empty squares to place 1 O, or  $C(7, 1) = 7$  ways to place the 1 O. Overall, there are

$$C(9, 2) \times C(7, 1)$$

possibilities.

**Answer 2:**

The tic-tac-toe board can be stored as a string of length 9, made up of 2 X's, 1 O, and 6 E's (for the empty spaces). For instance, if we read from left-to-right and top-to-bottom, the leftmost example above can be stored as the string  $XEEEOEEEX$ .

The number of permutations of this string is  $\frac{9!}{2!1!6!}$ .

